Trend Free Orthogonal Arrays using some Linear Codes

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Abstract : A method for constructing trend free orthogonal arrays using the parity check matrix of a linear code is proposed. This method can easily be used to construct the trend free orthogonal arrays of higher level and higher strength.

Key words: Orthogonal Arrays, Linear codes, Time count, Reed Muller code, BCH code.

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1. INTRODUCTION

One of the basic principles in experimentation is randomization of treatments. But randomization may not always give the efficient results. For instance, there may be an unknown or uncontrollable trend effect which is highly correlated with the order in which the observations are obtained. In such situations, one may prefer to assign treatments to experimental units in such a way that the usual estimates(main or interaction effects) for the factorial effects of interest are not affected by unknown trend. Such run orders are called **trend free run orders**. When trend free effects are considered in factorial experiments, the order of experimental runs is essential and one may use ***orthogonal arrays*** in such situations. Orthogonal Arrays introduced by [10], [11], are of importance because of their role in experimental designs as universally optimal fractional factorials. Orthogonal arrays have gained a renewed interest in industrial experimentation for product improvement, mainly after the work by [18] and his colleagues. Asymmetric Orthogonal arrays also introduced by [12] , have been used extensively in industrial experiments for quality improvement and their use in other experimental situations has also been widespread. These arrays are closely related to combinatorics, finite fields, finite geometry and error-correcting codes. For details see [7] . To construct the trend free run orders for orthogonal arrays, one needs to derive the trend free property in the columns of an array to gain an appropriate order. [17], [15], [16] ,[1] proposed algorithms to construct trend free run orders of orthogonal arrays. [13] constructed trend free orthogonal arrays using a class of formally self dual linear codes given by [3].

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. In this paper, we present a systematic method to construct trend free run orders for orthogonal arrays using the parity check matrices of linear codes and the results given by [4]. The Trend free orthogonal arrays are constructed using Reed Muller , Cyclic , BCH, MDS and Golay codes.

The paper is organized as follows. Section 2 gives the preliminaries required. Section 3 gives a brief description of coding theory. The construction technique for trend free symmetric and asymmetric orthogonal arrays is presented in Section 4 and Section 5 respectively.

2. PRELIMINARIES

Definition 1: An Orthogonal Array OA(N, n, $q_1 \times q_2 \times \ldots\ldots \times q_n$, g) of strength g, $2 \le g \le n$ is an N×n matrix having q_i (≥2) distinct symbols in the i^{th} column, i =1,2,…,n such that in every N×g submatrix, all possible combinations of symbols appear equally often as a row. In particular, if $q_1 = \cdots\cdots = q_n = q$, the orthogonal array is called a symmetric orthogonal array and is denoted by OA(N, n, q, g) otherwise, the array is called asymmetric orthogonal array.

Definition 2.: Let $\mathbf{Y} = (y_1, y_2, \ldots y_N)'$ denotes the ordered vector of observations, and $T_x = (1^x, 2^x, \ldots N^x)'$ for x = 0,1,2,... v be the N × 1 vector of trend coefficients and let a_i

be the contrast for main effect A_i ; i=1,2,...n, in the run order. Then the quantity $a_i' T_x$ is known as the time count for main effect A_i ..

A necessary and sufficient condition for a main effect contrast a to be $v-$ trend free is that

$$a' \; T_x \; = 0 \quad \forall \; x = 0,1,2,,..,v \qquad (2.1)$$

In general, an N × 1 vector a is called v-trend free if (2.1) holds.

***Definition 3*:** A run order is optimal for the estimation of the factor effects of interest in the presence of nuisance v-degree polynomial trend if

$$\boldsymbol{X'T = 0} \qquad (2.2)$$

Where **X** is an N×n matrix of factor effect coefficients and T is an N×v matrix of polynomial trend coefficients. If (2.2) is satisfied then the run order is said to be v trend free. If **x** is any column of **X** and **t** is any column of **T** then the usual inner product $\mathbf{x't}$ is called the time count between **x** and **t**. Criterion (2.2) states that all the time count are zero for optimal run order.

3. LINEAR CODES

A linear $[n,k,d]_q$ code C over GF(q), where q is a prime or a prime power, n is the length, k is the dimension and d is the minimum distance, is a k-dimensional subspace of the n-dimensional vector space V(n,q) over GF(q). The elements of C are called codewords. The minimum distance d of a code is the smallest number of positions in which two different codewords of C differ. Equivalently, d is the smallest number of nonzero symbols in any nonzero codeword of C. A linear code may be concisely specified by a k×n generator matrix G whose rows form a basis for the code. The standard form of the generator matrix is

$$G = [\; \mathbf{I_k} \mid B \;]$$

where B is a k × (n – k) matrix with entries from GF(q).

The dual code $C^\perp$ of the $[n,k,d]_q$ code C is an $[n,n-k,,d^\perp]_q$ where $C^\perp = \{ e \in V(n,q) \mid e.\theta = 0$ for all $\theta \in C\}$. This has an (n-k)×n generator matrix H which is called the parity check matrix of the code C. If the generator matrix is given in the standard form, the corresponding parity check matrix is given as

- $H = (-B^T_{n-k \times k} \mid I_{n-k})$

Any $d^\perp$-1 columns in generator matrix G of C are linearly independent and any d-1 columns in parity check matrix H are linearly independent.

4. TREND FREE RUN ORDERS FOR SYMMETRIC ORTHOGONAL ARRAYS

In this section, we construct trend free run orders for symmetric orthogonal arrays. [4] used Generalized Foldover Scheme (GFS) to construct trend free designs and also discussed conditions for linear trend free effects in GFS. These conditions involve the generator matrices. We give below the result of [4] as given in [6]

***Theorem 1*:** *Let q (≥2) be a prime or prime power. Suppose that there exists an h × n matrix* $\boldsymbol{M}$ *, with elements from GF(q), such that*

i) *Every h × g submatrix of* $\boldsymbol{M}$ *has rank g and*
ii) *Every column of* $\boldsymbol{M}$ *has at least (v+1) non zero elements.*

Then there exists a symmetric orthogonal array OA(q^h *, n , q ,g) in which all main effects are v-trend free.*

The method of construction in theorem 1 is known as **Generalised Foldover Technique.** The conditions (i) and (ii) in above theorem involve generator matrices. [4] provided a method for construction of generator matrices so that the systematic run order for a design is obtained using GFS. However, their method of construction of generator matrices is difficult to use. The generator matrices for the construction of systematic run order can also be obtained from linear codes. We now give a systematic method to construct trend free orthogonal arrays using a parity check matrix of linear code.

Method Of Construction :

I. Consider the parity check matrix H, of a linear $[n,k,d]_q$ code. It is an $(n\text{-}k)\times n$ matrix with elements from GF(q).

II. Obtain a matrix M by retaining the columns of H which have weight $\geq w\ (\geq 2)$, say where the weight of a column means the number of non-zero elements in the column. Let l denotes the number of columns with weight w. Then M is a matrix of order $(n\text{-}k)\times l$. Clearly the matrix M, satisfies the conditions of Theorem 1 with $h = n\text{-}k$, $n = l$, $g = d\text{-}1$ and $v = w\text{-}1$.

III. Let ξ denote a $(n\text{-}k)\times 1$ vector with enteries from GF(q). Consider all possible q^{n-k} distinct choices of ξ over GF(q) and write down $\xi'M$ one by one to obtain the orthogonal array OA(q^{n-k} ,l, q, g) in which all the main effects are $v = w\text{-}1$ trend free.

Above method of construction can be stated in the form of following theorem :

Theorem 2: *Existence of a linear $[n,k,d]_q$ code implies the existence of a symmetric orthogonal array (q^h, l, q, d-1) in which all main effects are (w-1)-trend free , l is the number of columns having weight $\geq w$ and h=n-k.*

We consider some linear codes and use Theorem 2 to construct orthogonal arrays from these. For details on these codes, see [9].

4.1 REED MULLER CODES :

The r^{th} order binary Reed-Muller code R(r,m) of length $n = 2^m$, for $0 \leq r \leq m$, is the set of all vectors f, where $f(j_1,\ldots,j_m)$ is a Boolean function which is a polynomial of degree at most r. For any m and any r, $0 \leq r \leq m$, there is a binary r^{th} order RM code R(r,m) with the following properties:

Length $n = 2^m$, dimension $k = 1+\binom{m}{1}+\ldots+\binom{m}{r}$ and minimum distance 2^{m-r} . The parity check matrix of RM(r,m) code can be expressed in the form $H = (-B^T|I_{n-k})$. The matrix B has all the columns with weight greater than or equal to 2, The dual code of RM(r,m) is RM(m-r-1,m). Any $2^{m-r}-1$ columns in parity check matrix H are linearly independent. Using theorem 2 we get OA($2^{2^{m}-k}, 2^m, 2,\ 2^{m-r}-1$). The method is explained in the following example.

Example *1*: Let r=2 and m=4. The parity check matrix of RM(2,4) is given as

$$H = \begin{bmatrix} 1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1 \\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0 \\ 1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0 \\ 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Retaining the columns of H which have weight $\geq$ 2, We get the following matrix M

$$M = \begin{bmatrix} 1&0&0&1&0&1&1&0&1&1&1 \\ 1&0&1&0&1&0&1&1&0&1&1 \\ 1&1&0&0&1&1&0&1&1&0&1 \\ 1&1&1&1&0&0&0&1&1&1&0 \\ 1&1&1&1&1&1&1&0&0&0&0 \end{bmatrix}$$

Any three columns in M are linearly independent. Hence the matrix M satisfy the conditions of Theorem 2, with q = 2, h=5, g = 3, w = 4, l=11. Let ξ denote a 5×1 vector with entries from GF(2). Considering all the 2^5 possible distinct choices of ξ as W (given below) and on computing $W'M$ we get $2^5\times 11$ array given in Table 1 which is an OA(2^5,11,2,3).

$$W = \begin{bmatrix} 0000000000000000\ 1111111111111111 \\ 000000001111111\,1\,0000000011111111 \\ 000011110000111\,1\,0000111100001111 \\ 001100110011001\,1\,0011001100110011 \\ 010101010101010\,1\,0101010101010101 \end{bmatrix}$$

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Table 1: Trend free OA(2^5,11,2,3) along with T.C.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	1	0	1	1	0	1	1	1
	1	0	1	0	1	0	1	1	0	1	1
	0	0	1	1	1	1	0	1	1	0	0
	1	1	0	0	1	1	0	1	1	0	1
	0	1	0	1	1	0	1	1	0	1	0
	0	1	1	0	0	1	1	0	1	1	0
	1	1	1	1	0	0	0	0	0	0	1
	1	1	1	1	0	0	0	1	1	1	0
	0	1	1	0	0	1	1	1	0	0	1
	0	1	0	1	1	0	1	0	1	0	1
	1	1	0	0	1	1	0	0	0	1	0
	0	0	1	1	1	1	0	0	0	1	1
	1	0	1	0	1	0	1	0	1	0	0
	1	0	0	1	0	1	1	1	0	0	0
	0	0	0	0	0	0	0	1	1	1	1
	1	1	1	1	1	1	1	0	0	0	0
	0	1	1	0	1	0	0	0	1	1	1
	0	1	0	1	0	1	0	1	0	1	1
	1	1	0	0	0	0	1	1	1	0	0
	0	0	1	1	0	0	1	1	1	0	1
	1	0	1	0	0	1	0	1	0	1	0
	1	0	0	1	1	0	0	0	1	1	0
	0	0	0	0	1	1	1	0	0	0	1
	0	0	0	0	1	1	1	1	1	1	0
	1	0	0	1	1	0	0	1	0	0	1
	1	0	1	0	0	1	0	0	1	0	1
	0	0	1	1	0	0	1	0	0	1	0
	1	1	0	0	0	0	1	0	0	1	1
	0	1	0	1	0	1	0	0	1	0	0
	0	1	1	0	1	0	0	1	0	0	0
	1	1	1	1	1	1	1	1	1	1	1
$T.C^1$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
$T.C^2$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

The array generated above forms an OA(2^5,11,2,3). Replacing all zero symbol in the array by -1's and multiplying with $T_1 = (1,2,3,\ldots,16)$ and $T_2 = (1^2,2^2,\ldots 16^2)$ for linear and quadratic trend effect respectively ,we get the time counts $\mathbf{T.C^1}$ and $\mathbf{T.C^2}$ and observe that these are zero for all the main effects. (Table 1). Hence, we get a trend free run order for OA(2^5,11,2,3) having all the main effects linear and quadratic trend free.

4.2 CYCLIC CODES :

A linear code over GF(q) is said to be cyclic if whenever $(c_0,c_1,\ldots,c_{n-2},c_{n-1})$ is a codeword so also is $(c_1,c_2,\ldots,c_{n-1},c_0)$. Cyclic arrays can be described by a single generating vector $z = (z_0 z_1 \ldots\ldots z_{n-1})$ such that the generator matrix consists of this vector and its first (k-1) cyclic shifts. The generating vector z is represented by a polynomial $z(x) = z_0 + z_1X + \ldots + z_{n-1}X^{n-1}$ which is called a generator polynomial for the code. If a code is cyclic, so is its dual, and the generator polynomial of its dual can be obtained by the following result given in [9].

Theorem 3 : *If C is a cyclic code of length n over GF(q), with generator polynomial z(x), then the dual code $C^{\perp}$ is also cyclic and has generator polynomial*

$$\mathbf{h}^*(\mathbf{x}) = \frac{\mathbf{X^n-1}}{\mathbf{z^*(x)}}$$

where $z^*(x) = X^{deg.z} z(x^{-1})$ is reciprocal polynomial to z(x).

Example 2: Let $z(x) = (x^4+x+1)(x^4+x^3+x^2+x+1) = 1+x^4+x^6+x^7$ be the generator polynomial for $(15,7,5)_2$ code. Then the generator polynomial for the dual of this code is given as

$$h(x) = x^7(1+x^{-4}+x^{-6}+x^{-7}) = 1+x+x^3+x^7$$

Hence the parity check matrix of $(15,7,5)_2$ code is obtained by writing the coefficients and giving the cyclic shift to the coefficients, as given below

$$H = \begin{bmatrix} 1 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 1 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 & 1 & 0 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 1 \end{bmatrix}$$

Selecting the columns with bold face (as the weight of the columns is at least 2), the generator matrix Z_1 is obtained in which any four columns are linearly independent. Thus q=2, h = 8, $v = w\text{-}1$ =1 , l =10 and g = 4. Using Theorem 2 we obtain an OA(2^8,10,2,4). In this array all main effects are at least linear trend free. The design is listed in table 2.

.3 BCH CODES :

The BCH codes over GF(q) of length $n = q^m - 1$ and designed distance δ is the largest possible cyclic code having zeroes $\alpha^b, \alpha^{b+1}, \ldots, \alpha^{b+\delta-2}$ where $\alpha \in GF(q^m)$ is the primitive n^{th} root of unity, b is a non negative integer and m is the multiplicative order of q mod n.

The parity check matrix for a BCH code with b=1 is given by

$$C = \begin{bmatrix} 1 & \alpha & \alpha^2 & \ldots\ldots\ldots\ \alpha^{n-1} \\ 1 & \alpha^3 & (\alpha^3)^2 & \ldots\ldots\ldots\ (\alpha^3)^{n-1} \\ 1 & \alpha^5 & (\alpha^5)^2 & \ldots\ldots\ldots\ (\alpha^5)^{n-1} \\ . & . & . & . \quad . \quad . \\ . & . & . & . \quad . \quad . \\ . & . & . & . \quad . \quad . \\ 1 & (\alpha^{\delta-2}) & (\alpha^{\delta-2})^2 & \ldots\ldots\ldots (\alpha^{\delta-2})^{n-1} \end{bmatrix}$$

where each entry is replaced by the corresponding binary m-tuple.

Example 3 : Let n =15, δ = 7. Then the matrix C of $(15, 5, 7)_2$ code is given as

$$C = \begin{bmatrix} 1&0&0&0&1&0&0&1&1&0&1&0&1&1&1 \\ 0&1&0&0&1&1&0&1&0&1&1&1&1&0&0 \\ 0&0&1&0&0&1&1&0&1&0&1&1&1&1&0 \\ 0&0&0&1&0&0&1&1&0&1&0&1&1&1&1 \\ 1&0&0&0&1&1&0&0&0&1&1&0&0&0&1 \\ 0&0&0&1&1&0&0&0&0&1&0&0&0&1&1 \\ 0&0&1&0&1&0&0&1&1&1&0&0&1&0&1 \\ 0&1&1&1&1&0&1&1&1&1&0&1&1&1&1 \\ 1&0&1&1&0&1&1&0&0&1&0&1&1&0&1 \\ 0&1&1&0&1&1&0&1&1&0&1&1&0&1&1 \\ \mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{1} \\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} \end{bmatrix}$$

Deleting the rows (with bold face) being the row of all zeroes and a repeated row, we get the parity check matrix Z_2 in which any six columns are linearly independent. This matrix satisfies the condition of theorem 2 with q=2, h=10, *w*=3, *l* = 15. The design obtained has all main effects at least linear trend free and is listed in table 2 .

4.4 MDS CODES :

A linear code $C[n, k, d]_q$ with d= n-k+1 is called maximum distance separable code. If $C[n,k,d]_q$ is MDS code, then the dual code $C^{\perp}$ is a linear $[n, n\text{-}k, k+1]_q$ MDS code. Every k columns of a generator matrix G are linearly independent or every (n-k) columns of parity check matrix are linearly independent. An $[n,k,d]_q$ code C with generator matrix G = [I | B], where B is a k × (n-k) matrix, is MDS iff every square submatrix (formed by any i rows and i columns) for any i = 1,2,……,min{k,n-k} of B is non singular.

Example 4: For q = 5 consider the parity check of $[8,4,5]_5$ code in which any four columns are independent

$$\boldsymbol{G_2} = \begin{bmatrix} 1&0&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1} \\ 0&1&0&0&\mathbf{1}&\mathbf{2}&\mathbf{3}&\mathbf{4} \\ 0&0&1&0&\mathbf{1}&\mathbf{3}&\mathbf{4}&\mathbf{0} \\ 0&0&0&1&\mathbf{1}&\mathbf{4}&\mathbf{0}&\mathbf{0} \end{bmatrix}$$

Retaining the columns with bold face i.e. with weight ≥ 2, we get the matrix Z_3 and using the method we obtain a symmetric orthogonal array with parameters q = 5, *h* = 4, g =4, *w* = 2, *l* = 4. The design obtained has all main effects linear trend free.and is listed in table 2.

Example 5: let q=7 , the parity check of $[12,6,7]_7$ MDS code, in which any six columns are linearly independent is given below

$$\boldsymbol{G_3} = \begin{bmatrix} 1&0&0&0&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1} \\ 0&1&0&0&0&0&\mathbf{1}&\mathbf{3}&\mathbf{6}&\mathbf{4}&\mathbf{2}&\mathbf{5} \\ 0&0&1&0&0&0&\mathbf{1}&\mathbf{6}&\mathbf{4}&\mathbf{2}&\mathbf{5}&\mathbf{0} \\ 0&0&0&1&0&0&\mathbf{1}&\mathbf{4}&\mathbf{2}&\mathbf{5}&\mathbf{0}&\mathbf{0} \\ 0&0&0&0&1&0&\mathbf{1}&\mathbf{2}&\mathbf{5}&\mathbf{0}&\mathbf{0}&\mathbf{0} \\ 0&0&0&0&0&1&\mathbf{1}&\mathbf{5}&\mathbf{0}&\mathbf{0}&\mathbf{0}&0 \end{bmatrix}$$

Taking the columns with weight ≥ 2, we get the generator matrix Z_4 , and using the method we obtain a symmetric orthogonal array with parameters q = 7, h = 6, g = 6, *w* = 2, *l* = 4. Further all main effects in the design are linear trend free.

4.5 TERNARY GOLAY CODE:

$[11,6,5]_3$ is a linear code over a ternary alphabet, the relative distance of the codes is as large as it possibly can be for a ternary code, and hence the ternary Golay code is a perfect code.

Example 6: The parity check matrix of $[11,6,5]_3$ Golay code is given as

$$H_1 = \begin{bmatrix} \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{2}&\mathbf{2}&\mathbf{0}&1&0&0&0&0 \\ \mathbf{1}&\mathbf{1}&\mathbf{2}&\mathbf{1}&\mathbf{0}&\mathbf{2}&0&1&0&0&0 \\ \mathbf{1}&\mathbf{2}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{2}&0&0&1&0&0 \\ \mathbf{1}&\mathbf{2}&\mathbf{0}&\mathbf{1}&\mathbf{2}&\mathbf{1}&0&0&0&1&0 \\ \mathbf{1}&\mathbf{0}&\mathbf{2}&\mathbf{2}&\mathbf{1}&\mathbf{1}&0&0&0&0&1 \end{bmatrix}$$

Selecting the columns (with bold face) i.e. weight ≥ 4, we get the matrix Z_5 in which any four columns are linearly independent and using the method we obtain a symmetric orthogonal array with parameters q = 3, h =5, *l* = 6 , g = 3, *w* = 4, Z_5 generates the design in which all main effects are 3-trend free.

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Table 2 lists the parameters of the trend free linear symmetric orthogonal arrays generated using Theorem 2 and above described Linear codes.

Table 2: Trend free Symmetric Orthogonal Arrays based on different codes

Type of the code	***Parameters of the code***	***Trend free symmetric orthogonal arrays***	***Degree of trend free for main effects in symmetric orthogonal arrays***
REED MULLER code	RM(2,4)	OA(2^5,11,2,3)	Linear*
Cyclic Code	$(15,7,5)_2$	OA(2^8,10,2,4)	Linear*
BCH code	$(15,5,7)_2$	OA(2^{10},15,2,5)	Linear*
MDS code	$(8,4,5)_5$ $(12,6,7)_7$	OA(5^4,4,5 4) OA(7^6,6,7,6)	Linear Linear
GOLAY code	$(11,6,5)_3$	OA(3^5, 6 ,3,4)	Linear*

***indicates** that main effects are trend free for higher order also.

5. TREND FREE RUN ORDERS FOR ASYMMETRIC ORTHOGONAL ARRAYS

Asymmetric orthogonal arrays are inevitable in many experimental situations and thus it is important to study the trend freeness property of asymmetric orthogonal arrays. Trend free run orders for asymmetric orthogonal arrays can also be obtained from the parity check matrix of a linear $[n,k,d]_q$ code. Using the matrices obtained from linear codes in Section 4, trend free run orders for asymmetric orthogonal arrays can be constructed.

Consider an OA(N, l, $q_1 \times q_2 \times \ldots\ldots \times q_n$, g) whose columns are called as factors denoted by $F_1, F_2, \ldots, F_l$. Also consider GF(q), of order q, where q is a prime or prime power. For the factor F_i $(1 \le i \le l)$ define u_i columns, say $p_{i_1}, p_{i_2}, \ldots, p_{i_{u_i}}$, each of order h $\times$ 1 with elements from GF(q). Thus for the l factors we have in all $\sum_{i=1}^{l} u_i$ columns

We state the theorem proved in [2] to construct an asymmetric orthogonal array

Theorem 4: *Let M be the h $\times l$ matrix, where $l = \sum u_{i_j}$ and h $\ge \sum u_{i_j}$ such that any d –1 columns of M are linearly independent. Then M can be partitioned as M = $[A_1\ A_2 \ldots A_l]$, where $A_i = [p_{i_1}\ p_{i_2} \ldots p_{i_{u_i}}]$, $1 \le i \le l$. Then for each of the matrices $[A_{i_1}\ A_{i_2} \ldots A_{i_g}]$; where $g \le d-2$, out of $A_1\ A_2 \ldots A_l$ the h $\times \sum u_{i_j}$ matrix $[A_{i_1}\ A_{i_2} \ldots A_{i_g}]$ has full column rank over GF(q), Then an $OA(q^h, l, q^{u_1} \times q^{u_2} \times \ldots q^{u_l}, g)$, can be constructed.*

Using Theorem 4 with the construction technique we state the following theorem for construction of trend free asymmetric orthogonal arrays using parity check matrix of linear code.

Theorem 5: *Existence of a linear $[n, k, d]_q$ code implies the existence of an asymmetric $OA(q^h, l, q^{u_1} \times q^{u_2} \times \ldots q^{u_l}, g)$, where h= n-k, $g \le d$-2 and l is the number of columns with weight $\ge w$, in which all main effects are (w-1)-trend free.*

The technique of construction can be explained with the help of following example.

Example 7: Consider the matrix M obtained in Example 1. Represent M as $[M_1\ M_2 \ldots M_{11}]$, where M_i; $1 \le i \le 11$ denotes the i^{th} column of matrix M.

To construct an orthogonal array OA(2^5,10,(2^2) × 2^9, 2) we choose the following matrices, corresponding to the factors of the array.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = [\, M_1\ M_2\,], \quad A_i = M_{i+1} \quad 2 \le i \le 9$$

The condition of the Theorem 4 is always satisfied for g = 2 by the above matrix M. This can also be shown with above choices of A_i matrices corresponding to the 10 factors.

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(i) Let i = 1 and j ∈ { 2, 3, …, 10} ; i ≠ j. For this choice of indices i, j the matrix $[A_i, A_j]$ will always have rank 2 since any 3 or fewer columns of M are linearly independent.

(ii) Let i, j ∈ { 2, 3, ... , 10} ; i ≠ j. For this choice of the indices i and j, the matrix $[A_i, A_j]$ will always have rank 2, because any 3 columns of the matrix M are linearly independent.

Thus in each case the conditions of Theorem 4 are satisfied and the desired orthogonal array can be constructed by Computing W′M where W is possible distinct choices of ξ and ξ is a $2^5 \times 1$vector with enteries from GF(2). Further replacing the 4 combinations (00), (01), (10), (11) under the first two columns by 4 distinct symbols 0, 1, 2, 3 respectively we get an OA(2^5 ,10,$(2^2) \times 2^9$, 2) shown in Table 3. Here we observe that the run order for column (factor) A_1 with 4 symbols (with bold face) is also linear trend free (as time count = 0) with the other remaining nine columns(factors), A_i i=2,3…,10 Thus we get trend free run order for asymmetric orthogonal array OA(2^5 ,10,$(2^2) \times 2^9$, 2) in which all the main effects are linear trend free.

Table 4 lists all the possible linear trend free asymmetric orthogonal arrays generated using theorem 4 and the codes mentioned in section 4.

Table 3: Linear Trend free OA(2^5 ,10,$(2^2) \times 2^9$, 2)

		A_1	**A_2**	**A_3**	**A_4**	**A_5**	**A_6**	**A_7**	**A_8**	**A_9**	**A_{10}**
	00→	***0***	0	0	0	0	0	0	0	0	0
	10→	***2***	0	1	0	1	1	0	1	1	1
	10→	***2***	1	0	1	0	1	1	0	1	1
	00→	***0***	1	1	1	1	0	1	1	0	0
	11→	***3***	0	0	1	1	0	1	1	0	1
	01→	***1***	0	1	1	0	1	1	0	1	0
	01→	***1***	1	0	0	1	1	0	1	1	0
	11→	***3***	1	1	0	0	0	0	0	0	1
	11→	***3***	1	1	0	0	0	1	1	1	0
	01→	***1***	1	0	0	1	1	1	0	0	1
	01→	***1***	0	1	1	0	1	0	1	0	1
	11→	***3***	0	0	1	1	0	0	0	1	0
	00→	***0***	1	1	1	1	0	0	0	1	1
	00→	***2***	1	0	1	0	1	0	1	0	0
	00→	***2***	0	1	0	1	1	1	0	0	0
	00→	***0***	0	0	0	0	0	1	1	1	1
	11→	***3***	1	1	1	1	1	0	0	0	0
	11→	***1***	1	0	1	0	0	0	1	1	1
	11→	***1***	0	1	0	1	0	1	0	1	1
	11→	***3***	0	0	0	0	1	1	1	0	0
	00→	***0***	1	1	0	0	1	1	1	0	1
	10→	***2***	1	0	0	1	0	1	0	1	0
	10→	***2***	0	1	1	0	0	0	1	1	0
	00→	***0***	0	0	1	1	1	0	0	0	1
	00→	***0***	0	0	1	1	1	1	1	1	0
	10→	***2***	0	1	1	0	0	1	0	0	1
	10→	***2***	1	0	0	1	0	0	1	0	1
	00→	***0***	1	1	0	0	1	0	0	1	0
	11→	***3***	0	0	0	0	1	0	0	1	1
	01→	***1***	0	1	0	1	0	0	1	0	0
	01→	***1***	1	0	1	0	0	1	0	0	0
	11→	***3***	1	1	1	1	1	1	1	1	1
T.C		**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Table 4: Trend free Asymmetric Orthogonal Arrays based On different codes

Type of the code	*Parameters of the code*	*Trend free symmetric orthogonal arrays*	*Degree of trend free for main effects in Asymmetric orthogonal arrays*
Cyclic Code	$(15,7,5)_2$	$(2^5,10,(2^2)\times 2^9,3)$ $(2^5,9,(2^2)^2\times 2^7,2)$	Linear* [1] [2]
BCH code	$(15,5,7)_2$	$(2^{10},4,(2^2)\times 2^3,4)$ $(2^{10},3,(2^2)^2\times 2,3)$ $(2^{10},2,(2^3)\times 2^2,2)$	Linear* [3]
MDS code	$(8,4,5)_5$	$(5^4,3,(5^2)\times 5^2,2)$	Linear
MDS code	$(12,6,7)_7$	$(7^6,5,(7^2)\times 7^4,5)$ $(7^6,4,(7^2)^2\times 7,4)$ $(7^6,4,(7^3)\times 7^3,4)$	Linear
REED MULLER code	RM(2,4)	$(2^5,10,(2^2)\times 2^9,2)$	Linear* [7] [8]
GOLAY code	$(11,6,5)_3$	$(3^5,5,(3^2)\times 3^4,3)$ $(3^5,4,(3^2)^2\times 3^2,2)$ $(3^5,4,(3^3)\times 3^3,2)$	Linear* [9]

*****Indicates that the designs are trend free for higher order also

CONCLUSION: We give a systematic method to construct trend free symmetric and asymmetric orthogonal arrays using the parity check matrix of linear code. This method can be used easily to generate the arrays with higher level in which all main effects are with higher degree of trend freeness .

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